The group of linear isometries of the Gurarij space is extremely amenable

Dana Bartošová [1]
Jordi López-Abad [2]   Brice R. Mbombo [3]

[1,3]University of São Paulo

[2]ICMAT Madrid

When Topological Dynamics meets Model Theory
July 1

The first author was supported by the grants FAPESP 2013/14458-9 and FAPESP 2014/12405-8.

Outline

(EA) Extreme amenability

- and a connection to Ramsey theory.

Outline

(EA) Extreme amenability

- and a connection to Ramsey theory.

($\mathbb{G}$) Gurarij space

- group of linear isometries
- approximate Ramsey propety for finite dimensional normed spaces

Outline

(EA) Extreme amenability
- and a connection to Ramsey theory.

($\mathbb{G}$) Gurarij space
- group of linear isometries
- approximate Ramsey propety for finite dimensional normed spaces

(S) Poulsen simplex
- new characterization
- group of affine homeomorphisms
- approximate Ramsey propety for finite dimensional simplexes

Outline

(EA) Extreme amenability
- and a connection to Ramsey theory.

($\mathbb{G}$) Gurarij space
- group of linear isometries
- approximate Ramsey propety for finite dimensional normed spaces

(S) Poulsen simplex
- new characterization
- group of affine homeomorphisms
- approximate Ramsey propety for finite dimensional simplexes

(M) Miscellaneous
- Hilbert cube
- Pseudoarc

Extremely amenable groups

A topological group G is extremely amenable every G-action on a compact Hausdorff space X has a fixed point, that is, $x \in X$ s.t. $gx = x$ for all $g \in G$.

Extremely amenable groups

A topological group G is extremely amenable every G-action on a compact Hausdorff space X has a fixed point, that is, $x \in X$ s.t. $gx = x$ for all $g \in G$.

Examples (Pestov)

- $\mathrm{Homeo}_+([0,1])$

Extremely amenable groups

A topological group G is extremely amenable every G-action on a compact Hausdorff space X has a fixed point, that is, $x \in X$ s.t. $gx = x$ for all $g \in G$.

Examples (Pestov)

- $\mathrm{Homeo}_+([0,1])$
- $\mathrm{Aut}(\mathbb{Q}, <)$

Extremely amenable groups

A topological group G is extremely amenable every G-action on a compact Hausdorff space X has a fixed point, that is, $x \in X$ s.t. $gx = x$ for all $g \in G$.

Examples (Pestov)

- $\mathrm{Homeo}_+([0,1])$
- $\mathrm{Aut}(\mathbb{Q}, <)$
- $\mathrm{Iso}(\mathbb{U}, d)$

Extremely amenable groups

A topological group G is extremely amenable every G-action on a compact Hausdorff space X has a fixed point, that is, $x \in X$ s.t. $gx = x$ for all $g \in G$.

Examples (Pestov)

- $\mathrm{Homeo}_+([0,1])$
- $\mathrm{Aut}(\mathbb{Q}, <)$
- $\mathrm{Iso}(\mathbb{U}, d)$

FIRST

"Exotic groups" (Herrer–Christensen)

Extremely amenable groups

A topological group G is extremely amenable every G-action on a compact Hausdorff space X has a fixed point, that is, $x \in X$ s.t. $gx = x$ for all $g \in G$.

Examples (Pestov)

- $\mathrm{Homeo}_+([0,1])$
- $\mathrm{Aut}(\mathbb{Q}, <)$
- $\mathrm{Iso}(\mathbb{U}, d)$

FIRST

"Exotic groups" (Herrer–Christensen)

$U(l_2)$ (Gromov–Milman)

Extremely amenable groups

A topological group G is extremely amenable every G-action on a compact Hausdorff space X has a fixed point, that is, $x \in X$ s.t. $gx = x$ for all $g \in G$.

Examples (Pestov)

- $\mathrm{Homeo}_+([0,1])$
- $\mathrm{Aut}(\mathbb{Q}, <)$
- $\mathrm{Iso}(\mathbb{U}, d)$

FIRST
"Exotic groups" (Herrer–Christensen)
$U(l_2)$ (Gromov–Milman)

Lemma (Bodirsky–Pinsker–Tsankov)

Open subgroup of an extremely amenable group is extremely amenable.

A (countable) structure $\mathcal{A}$ is ultrahomogeneous $\longleftrightarrow$ every partial finite isomorphism can be extended to an automorphism of $\mathcal{A}$.

Connections with Ramsey theory

A (countable) structure $\mathcal{A}$ is ultrahomogeneous $\longleftrightarrow$ every partial finite isomorphism can be extended to an automorphism of $\mathcal{A}$.

Theorem (KPT; NvT)

$\mathrm{Aut}(\mathcal{A})$ *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures of* $\mathcal{A}$ *satisfy the Ramsey property and are rigid.*

Connections with Ramsey theory

A (countable) structure $\mathcal{A}$ is ultrahomogeneous $\longleftrightarrow$ every partial finite isomorphism can be extended to an automorphism of $\mathcal{A}$.

Theorem (KPT; NvT)

$\mathrm{Aut}(\mathcal{A})$ *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures of* $\mathcal{A}$ *satisfy the Ramsey property and are rigid.*

Examples

- $(\mathbb{Q}, <)$
- finite linear orders (Ramsey)

Connections with Ramsey theory

A (countable) structure $\mathcal{A}$ is ultrahomogeneous $\longleftrightarrow$ every partial finite isomorphism can be extended to an automorphism of $\mathcal{A}$.

Theorem (KPT; NvT)

$\mathrm{Aut}(\mathcal{A})$ *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures of* $\mathcal{A}$ *satisfy the Ramsey property and are rigid.*

Examples

- $(\mathbb{Q}, <)$
- $(\mathcal{R}, <)$

- finite linear orders (Ramsey)
- finite linearly ordered graphs (NR; AH)

Connections with Ramsey theory

A (countable) structure $\mathcal{A}$ is ultrahomogeneous $\longleftrightarrow$ every partial finite isomorphism can be extended to an automorphism of $\mathcal{A}$.

Theorem (KPT; NvT)

$\mathrm{Aut}(\mathcal{A})$ *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures of* $\mathcal{A}$ *satisfy the Ramsey property and are rigid.*

Examples

- $(\mathbb{Q}, <)$
- $(\mathcal{R}, <)$
- $(C, \mathcal{C})$

- finite linear orders (Ramsey)
- finite linearly ordered graphs (NR; AH)
- finite Boolean algebras (GR)

and approximate Ramsey theory

Theorem (Melleray-Tsankov)

For M approximately ultrahomogeneous, $\mathrm{Iso}(M)$ *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures satisfy the approximate Ramsey property.*

Theorem (Melleray-Tsankov)

For M approximately ultrahomogeneous, $\mathrm{Iso}(M)$ *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures satisfy the approximate Ramsey property.*

Examples (B-LA-M)

- $\mathbb{G}$
- finitely-dimensional normed spaces

Theorem (Melleray-Tsankov)

For M approximately ultrahomogeneous, $\mathrm{Iso}(M)$ is extremely amenable $\longleftrightarrow$ finitely-generated substructures satisfy the approximate Ramsey property.

Examples (B-LA-M)

- $\mathbb{G}$
- (P, p)
- finitely-dimensional normed spaces
- finite-dimensional simplexes

Gurarij space $\mathbb{G}$

(1) separable Banach space

Gurarij space $\mathbb{G}$

(1) separable Banach space

(2) contains isometric copy of every finite dimensional Banach space

Gurarij space $\mathbb{G}$

(1) separable Banach space

(2) contains isometric copy of every finite dimensional Banach space

(3) for every E finite dimensional, $i : E \hookrightarrow \mathbb{G}$ isometric embedding and $\varepsilon > 0$ there is a linear isometry $f : \mathbb{G} \longrightarrow \mathbb{G}$

$$\|i - f \restriction E\| < \varepsilon$$

Gurarij space $\mathbb{G}$

(1) separable Banach space

(2) contains isometric copy of every finite dimensional Banach space

(3) for every E finite dimensional, $i : E \hookrightarrow \mathbb{G}$ isometric embedding and $\varepsilon > 0$ there is a linear isometry $f : \mathbb{G} \longrightarrow \mathbb{G}$

$$\|i - f \restriction E\| < \varepsilon$$

LUSKY

Conditions (1),(2),(3) uniquely define $\mathbb{G}$ up to a linear isometry.

Gurarij space $\mathbb{G}$

(1) separable Banach space

(2) contains isometric copy of every finite dimensional Banach space

(3) for every E finite dimensional, $i : E \hookrightarrow \mathbb{G}$ isometric embedding and $\varepsilon > 0$ there is a linear isometry $f : \mathbb{G} \longrightarrow \mathbb{G}$

$$\|i - f \restriction E\| < \varepsilon$$

LUSKY

Conditions (1),(2),(3) uniquely define $\mathbb{G}$ up to a linear isometry.

KUBIŚ-SOLECKI; HENSON

Simple proof - metric Fraïssé theory.

$\mathrm{Iso}_l(\mathbb{G})$ + point-wise convergence topology = Polish group

$\mathrm{Iso}_l(\mathbb{G})$ + point-wise convergence topology = Polish group

BASIS

Group of linear isometries

$\mathrm{Iso}_l(\mathbb{G})$ + point-wise convergence topology = Polish group

BASIS

- E - finite dimensional subspace of $\mathbb{G}$

Group of linear isometries

$\mathrm{Iso}_l(\mathbb{G})$ + point-wise convergence topology = Polish group

BASIS

- E - finite dimensional subspace of $\mathbb{G}$
- $\varepsilon > 0$

Group of linear isometries

$\mathrm{Iso}_l(\mathbb{G})$ + point-wise convergence topology = Polish group

BASIS

- E - finite dimensional subspace of $\mathbb{G}$
- $\varepsilon > 0$

$$V_\varepsilon(E) = \{g \in \mathrm{Iso}(\mathbb{G}) : \|g \restriction E - \mathrm{id} \restriction E\| < \varepsilon\}$$

$\mathrm{Iso}_l(\mathbb{G})$ + point-wise convergence topology = Polish group

BASIS

- E - finite dimensional subspace of $\mathbb{G}$
- $\varepsilon > 0$

$$V_\varepsilon(E) = \{g \in \mathrm{Iso}(\mathbb{G}) : \|g \restriction E - \mathrm{id} \restriction E\| < \varepsilon\}$$

BEN YAACOV

$\mathrm{Iso}_l(\mathbb{G})$ is a universal Polish group.

Group of linear isometries

$\mathrm{Iso}_l(\mathbb{G})$ + point-wise convergence topology = Polish group

BASIS

- E - finite dimensional subspace of $\mathbb{G}$
- $\varepsilon > 0$

$$V_\varepsilon(E) = \{g \in \mathrm{Iso}(\mathbb{G}) : \|g \restriction E - \mathrm{id} \restriction E\| < \varepsilon\}$$

BEN YAACOV

$\mathrm{Iso}_l(\mathbb{G})$ is a universal Polish group.

Katětov construction

Approximate Ramsey property for l_∞^n's

Approximate Ramsey property for l_∞^n's

Theorem (B-LA-M)

$d \leq m$

Approximate Ramsey property for l_∞^n's

Theorem (B-LA-M)

$d \leq m$

r - *number of colours*

Theorem (B-LA-M)

$d \leq m$

r - *number of colours*

$\varepsilon > 0$

Approximate Ramsey property for l_∞^n's

Theorem (B-LA-M)

$d \leq m$

r - *number of colours*

$\varepsilon > 0$

$\exists n$

Approximate Ramsey property for l_∞^n's

Theorem (B-LA-M)

$d \leq m$

r - *number of colours*

$\varepsilon > 0$

$\exists n$

for every colouring $c : \mathrm{Emb}(l_\infty^d, l_\infty^n) \longrightarrow \{0, 1, \ldots, r-1\}$

Approximate Ramsey property for l_∞^n's

Theorem (B-LA-M)

$d \leq m$

r - *number of colours*

$\varepsilon > 0$

$\exists n$

for every colouring $c : \mathrm{Emb}(l_\infty^d, l_\infty^n) \longrightarrow \{0, 1, \ldots, r-1\}$

there is $i \in \mathrm{Emb}(l_\infty^m, l_\infty^n)$ *and* $\alpha < r$

Approximate Ramsey property for l_∞^n's

Theorem (B-LA-M)

$d \leq m$

r - *number of colours*

$\varepsilon > 0$

$\exists n$

for every colouring $c : \mathrm{Emb}(l_\infty^d, l_\infty^n) \longrightarrow \{0, 1, \ldots, r-1\}$

there is $i \in \mathrm{Emb}(l_\infty^m, l_\infty^n)$ *and* $\alpha < r$

$$i \circ \mathrm{Emb}(l_\infty^d, l_\infty^m) \subset (c^{-1}(\alpha))_\varepsilon$$

Approximate Ramsey property for l_∞^n's

Theorem (B-LA-M)

$d \leq m$

r - *number of colours*

$\varepsilon > 0$

$\exists n$

for every colouring $c : \mathrm{Emb}(l_\infty^d, l_\infty^n) \longrightarrow \{0, 1, \ldots, r-1\}$

there is $i \in \mathrm{Emb}(l_\infty^m, l_\infty^n)$ *and* $\alpha < r$

$$i \circ \mathrm{Emb}(l_\infty^d, l_\infty^m) \subset (c^{-1}(\alpha))_\varepsilon$$

Theorem (B-LA-M)

$\mathrm{Iso}(\mathbb{G})$ *is extremely amenable.*

Pestov's characterization of extreme amenability

G - topological group

G - topological group
$f : G \longrightarrow \mathbb{R}$ is finitely oscillation stable if

G - topological group
$f : G \longrightarrow \mathbb{R}$ is finitely oscillation stable if $\forall X \subset G$ finite and $\varepsilon > 0$

G - topological group
$f : G \longrightarrow \mathbb{R}$ is finitely oscillation stable if $\forall X \subset G$ finite and $\varepsilon > 0 \;\; \exists g \in G$ such that $\operatorname{osc}(f \restriction gX) < \varepsilon$.

Pestov's characterization of extreme amenability

G - topological group
$f : G \longrightarrow \mathbb{R}$ is finitely oscillation stable if $\forall X \subset G$ finite and $\varepsilon > 0 \;\; \exists g \in G$ such that $\operatorname{osc}(f \restriction gX) < \varepsilon$.

Lemma (Pestov)

TFAE

- *G is extremely amenable,*
- *every $f : G \longrightarrow \mathbb{R}$ bounded left-uniformly continuous is finite oscillation stable.*

G - topological group
$f : G \longrightarrow \mathbb{R}$ is finitely oscillation stable if $\forall X \subset G$ finite and $\varepsilon > 0$ $\exists g \in G$ such that $\operatorname{osc}(f \restriction gX) < \varepsilon$.

Lemma (Pestov)

TFAE

- *G is extremely amenable,*
- *every $f : G \longrightarrow \mathbb{R}$ bounded left-uniformly continuous is finite oscillation stable.*

Dual Ramsey Theorem

Theorem (Graham and Rothschild)

For every $k \leq m$ and $r \geq 2$, there exists n such that for every colouring of the k-element partitions of n by r-many colours there is an m-element partition X of n such that all k-element coarsenings of X have the same colour.

Approximate Ramsey property for finite-dimensional normed spaces

E, F - finite dimensional spaces
$\theta \geq 1$

$$\mathrm{Emb}_\theta(E,F) = \{T : E \longrightarrow F : T \text{ embedding } \& \, \|\mathrm{T}\| \, \|\mathrm{T}^{-1}\| \leq \theta\}$$

Approximate Ramsey property for finite-dimensional normed spaces

E, F - finite dimensional spaces
$\theta \geq 1$

$$\mathrm{Emb}_\theta(E,F) = \{T : E \longrightarrow F : T \text{ embedding } \& \, \|\mathrm{T}\| \left\|\mathrm{T}^{-1}\right\| \leq \theta\}$$

Theorem (B-LA-M)

r - number of colours, $\varepsilon > 0 \longrightarrow \exists H$ f.d. with $\mathrm{Emb}(F,H) \neq \emptyset$ such that for every

$$c : \mathrm{Emb}_\theta(E,H) \longrightarrow \{0,1,\ldots,r-1\}$$

$\exists i \in \mathrm{Emb}_\theta(F,H)$ and $\alpha < r$ such that

$$i \circ \mathrm{Emb}_\theta(E,F) \subset (c^{-1}(\alpha))_{\theta-1+\varepsilon}$$

Urysohn space $\mathbb{U}$

Theorem

Finite metric spaces satisfy the approximate Ramsey property.

Theorem

Finite metric spaces satisfy the approximate Ramsey property.

Corollary (Pestov)

$\mathrm{Iso}(\mathbb{U})$ *is extremely amenable.*

Theorem

Finite metric spaces satisfy the approximate Ramsey property.

Corollary (Pestov)

$\mathrm{Iso}(\mathbb{U})$ *is extremely amenable.*

Theorem (Nešetřil)

Linearly ordered finite metric spaces satisfy the (exact) Ramsey property.

Poulsen simplex P

(1) metrizable

Poulsen simplex P

(1) metrizable
(2) contains every metrizable simplex as its face

Poulsen simplex P

(1) metrizable
(2) contains every metrizable simplex as its face
(3) for every two faces E, F of P with the same finite dimension, there is an affine autohomeomorphism of P mapping E onto F

Poulsen simplex P

(1) metrizable
(2) contains every metrizable simplex as its face
(3) for every two faces E, F of P with the same finite dimension, there is an affine autohomeomorphism of P mapping E onto F

LINDENSTRAUSS-OLSEN-STERNFELD
Properties (1),(2) and (3) uniquely determine P up to an affine homeomorphism.

Poulsen simplex P

(1) metrizable

(2) contains every metrizable simplex as its face

(3) for every two faces E, F of P with the same finite dimension, there is an affine autohomeomorphism of P mapping E onto F

LINDENSTRAUSS-OLSEN-STERNFELD

Properties (1),(2) and (3) uniquely determine P up to an affine homeomorphism.

POULSEN

The set of extreme points of P is dense in P.

Poulsen simplex P

(1) metrizable

(2) contains every metrizable simplex as its face

(3) for every two faces E, F of P with the same finite dimension, there is an affine autohomeomorphism of P mapping E onto F

LINDENSTRAUSS-OLSEN-STERNFELD

Properties (1),(2) and (3) uniquely determine P up to an affine homeomorphism.

POULSEN

The set of extreme points of P is dense in P.

FACT

$T : \{0,1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$ the shift $\Rightarrow$ T-invariant probability measures form P

A projective characterization of P

$S_n :=$ positive part of the unit ball of l_1^n – finite-dimensional simplex with $n+1$ extreme points

A projective characterization of P

$S_n :=$ positive part of the unit ball of l_1^n – finite-dimensional simplex with $n+1$ extreme points

$\mathrm{Epi}(S_n, S_m) :=$ continuous affine surjections $S_n \longrightarrow S_m$

A projective characterization of P

S_n := positive part of the unit ball of l_1^n – finite-dimensional simplex with $n+1$ extreme points

$\mathrm{Epi}(S_n, S_m)$:= continuous affine surjections $S_n \longrightarrow S_m$

$AH(P)$:= group of affine homeomorphisms of P + compact-open topology

A projective characterization of P

$S_n :=$ positive part of the unit ball of l_1^n – finite-dimensional simplex with $n+1$ extreme points

$\mathrm{Epi}(S_n, S_m) :=$ continuous affine surjections $S_n \longrightarrow S_m$

$AH(P) :=$ group of affine homeomorphisms of P + compact-open topology

(U) $\forall n \ \exists \phi : P \longrightarrow S_n$ – continuous affine surjection

(APU) $\forall \varepsilon > 0 \ \forall n \ \forall \phi_1, \phi_2 : P \longrightarrow S_n \ \exists f \in AH(P)$ with $d(\phi_1, \phi_2 \circ f) < \varepsilon$

A projective characterization of P

S_n := positive part of the unit ball of l_1^n – finite-dimensional simplex with $n+1$ extreme points

$\mathrm{Epi}(S_n, S_m)$:= continuous affine surjections $S_n \longrightarrow S_m$

$AH(P)$:= group of affine homeomorphisms of P + compact-open topology

(U) $\forall n\ \exists \phi : P \longrightarrow S_n$ – continuous affine surjection

(APU) $\forall \varepsilon > 0\ \forall n\ \forall \phi_1, \phi_2 : P \longrightarrow S_n\ \exists f \in AH(P)$ with $d(\phi_1, \phi_2 \circ f) < \varepsilon$

Theorem (B-LA-M)

(U) + (APU) characterize P among non-trivial metrizable simplexes up to affine homeomorphism.

Approximate Ramsey property for P

$\mathrm{Epi}_0(S_n, S_m)$ - continuous affine surjections preserving 0

Approximate Ramsey property for P

$\mathrm{Epi}_0(S_n, S_m)$ - continuous affine surjections preserving 0

Theorem (B-LA-M)

$d \leq m$ and r natural numbers and $\varepsilon > 0$ given $\longrightarrow$ $\exists n$ such that for every colouring

$$c : \mathrm{Epi}_0(S_n, S_d) \longrightarrow \{0, 1, \ldots, r\}$$

there is $\pi \in \mathrm{Epi}_0(S_n, S_m)$ and $\alpha < r$ such that

$$\mathrm{Epi}_0(S_m, S_d) \circ \pi \subset (c^{-1}(\alpha))_\varepsilon$$

Approximate Ramsey property for P

$\mathrm{Epi}_0(S_n, S_m)$ - continuous affine surjections preserving 0

Theorem (B-LA-M)

$d \leq m$ and r natural numbers and $\varepsilon > 0$ given $\longrightarrow$ $\exists n$ such that for every colouring

$$c : \mathrm{Epi}_0(S_n, S_d) \longrightarrow \{0, 1, \ldots, r\}$$

there is $\pi \in \mathrm{Epi}_0(S_n, S_m)$ and $\alpha < r$ such that

$$\mathrm{Epi}_0(S_m, S_d) \circ \pi \subset (c^{-1}(\alpha))_\varepsilon$$

p - extreme point of P
$AH_p(P) = \{f \in AH(P) : f(p) = p\}$

Approximate Ramsey property for P

$\mathrm{Epi}_0(S_n, S_m)$ - continuous affine surjections preserving 0

Theorem (B-LA-M)

$d \leq m$ and r natural numbers and $\varepsilon > 0$ given $\longrightarrow$ $\exists n$ such that for every colouring

$$c : \mathrm{Epi}_0(S_n, S_d) \longrightarrow \{0, 1, \ldots, r\}$$

there is $\pi \in \mathrm{Epi}_0(S_n, S_m)$ and $\alpha < r$ such that

$$\mathrm{Epi}_0(S_m, S_d) \circ \pi \subset (c^{-1}(\alpha))_\varepsilon$$

p - extreme point of P
$AH_p(P) = \{f \in AH(P) : f(p) = p\}$

Theorem (B-LA-M)

$AH_p(P)$ is extremely amenable.

Universal minimal flows

$G = \mathrm{Aut}(\mathcal{A})$ – $\mathcal{A}$ ultrahomogeneous

Universal minimal flows

$G = \mathrm{Aut}(\mathcal{A})$ – $\mathcal{A}$ ultrahomogeneous
$G^* = \mathrm{Aut}(\mathcal{A}^*)$ – $\mathcal{A}^*$ ultrahomogeneous expansion of $\mathcal{A}$

Universal minimal flows

$G = \mathrm{Aut}(\mathcal{A})$ – $\mathcal{A}$ ultrahomogeneous
$G^* = \mathrm{Aut}(\mathcal{A}^*)$ – $\mathcal{A}^*$ ultrahomogeneous expansion of $\mathcal{A}$
Finite substructures of $\mathcal{A}^*$ satisfy the Ramsey property and are rigid.

Universal minimal flows

$G = \mathrm{Aut}(\mathcal{A})$ – $\mathcal{A}$ ultrahomogeneous
$G^* = \mathrm{Aut}(\mathcal{A}^*)$ – $\mathcal{A}^*$ ultrahomogeneous expansion of $\mathcal{A}$
Finite substructures of $\mathcal{A}^*$ satisfy the Ramsey property and are rigid.

OFTEN $M(G) \cong \widehat{G/G^*}$

$G = \mathrm{Aut}(\mathcal{A})$ – $\mathcal{A}$ ultrahomogeneous
$G^* = \mathrm{Aut}(\mathcal{A}^*)$ – $\mathcal{A}^*$ ultrahomogeneous expansion of $\mathcal{A}$
Finite substructures of $\mathcal{A}^*$ satisfy the Ramsey property and are rigid.

OFTEN $M(G) \cong \widehat{G/G^*}$

Structure $\mathcal{A}$	$M(\mathrm{Aut}(\mathcal{A}))$	authors
S^1_+	S^1_+	Pestov
$\mathbb{N}$	linear orders on $\mathbb{N}$	Glasner and Weiss
random graph $\mathcal{R}$	linear orders on $\mathcal{R}$	KPT
Cantor space C	maximal chains of closed subsets of C	Glasner and Weiss
Lelek fan L	$\widehat{\mathrm{Homeo}(L)/\mathrm{Homeo}(L_<)}$	B-Kwiatkowska

Universal minimal flow of $AH(P)$

Theorem (B-LA-M)

$M(AH(P)) \cong \widehat{AH(P)/AH_p}(P) \cong P$

PROBLEM

What is the universal minimal flow of Homeo($\mathcal{Q}$)?

PROBLEM
What is the universal minimal flow of Homeo($\mathcal{Q}$)?

$\mathcal{Q}$ is homeomorphic to P.

Hilbert cube $\mathcal{Q} = [-1, 1]^{\mathbb{N}}$

PROBLEM
What is the universal minimal flow of Homeo($\mathcal{Q}$)?

$\mathcal{Q}$ is homeomorphic to P.

Theorem (B-LA-M)

Homeo($\mathcal{Q}$) *admits a closed subgroup with the universal minimal flow being the natural action on* $\mathcal{Q}$.

PROBLEM

What is the universal minimal flow of Homeo($\mathcal{Q}$)?

$\mathcal{Q}$ is homeomorphic to P.

Theorem (B-LA-M)

Homeo($\mathcal{Q}$) *admits a closed subgroup with the universal minimal flow being the natural action on* $\mathcal{Q}$.

$\mathcal{Q}$ with its natural convex structure.

Hilbert cube $\mathcal{Q} = [-1, 1]^{\mathbb{N}}$

PROBLEM
What is the universal minimal flow of Homeo($\mathcal{Q}$)?

$\mathcal{Q}$ is homeomorphic to P.

Theorem (B-LA-M)

Homeo($\mathcal{Q}$) *admits a closed subgroup with the universal minimal flow being the natural action on* $\mathcal{Q}$.

$\mathcal{Q}$ with its natural convex structure.

Theorem (B-LA-M)

Aut($\mathcal{Q}$) *is topologically isomorphic to* $\{-1, 1\}^{\mathbb{N}} \times S_\infty$.

Hilbert cube $\mathcal{Q} = [-1, 1]^{\mathbb{N}}$

PROBLEM
What is the universal minimal flow of Homeo($\mathcal{Q}$)?

$\mathcal{Q}$ is homeomorphic to P.

Theorem (B-LA-M)

Homeo($\mathcal{Q}$) *admits a closed subgroup with the universal minimal flow being the natural action on* $\mathcal{Q}$.

$\mathcal{Q}$ with its natural convex structure.

Theorem (B-LA-M)

Aut($\mathcal{Q}$) *is topologically isomorphic to* $\{-1, 1\}^{\mathbb{N}} \times S_\infty$.

Theorem (B-LA-M)

$M(\mathrm{Aut}(\mathcal{Q})) = \{-1, 1\}^{\mathbb{N}} \times LO(\mathbb{N})$.

Pseudo-arc P

Conjecture (Uspenskij)

The universal minimal flow of $\mathrm{Homeo}(P)$ *is its natural action on the pseudoarc.*

Conjecture (Uspenskij)

The universal minimal flow of $\mathrm{Homeo}(P)$ *is its natural action on the pseudoarc.*

IRWIN-SOLECKI

$(\mathbb{P}, E)$ - $\mathbb{P}$ the Cantor set, E closed edge relation and $\mathbb{P}/E \cong P$

Conjecture (Uspenskij)

The universal minimal flow of $\mathrm{Homeo}(P)$ *is its natural action on the pseudoarc.*

IRWIN-SOLECKI

$(\mathbb{P}, E)$ - $\mathbb{P}$ the Cantor set, E closed edge relation and $\mathbb{P}/E \cong P$

$\mathrm{Aut}(\mathbb{P}) \longrightarrow \mathrm{Homeo}(P)$ continuous with dense image.

Conjecture (Uspenskij)

The universal minimal flow of $\mathrm{Homeo}(P)$ *is its natural action on the pseudoarc.*

IRWIN-SOLECKI

$(\mathbb{P}, E)$ - $\mathbb{P}$ the Cantor set, E closed edge relation and $\mathbb{P}/E \cong P$

$\mathrm{Aut}(\mathbb{P}) \longrightarrow \mathrm{Homeo}(P)$ continuous with dense image.

Lemma (B-Kwiatkowska; Solecki)

$M(\mathrm{Aut}(\mathbb{P}))$ *is not metrizable.*

Lionel's conjecture

Oligomorphic automorphism groups of countable structures have metrizable universal minimal flows.

Oligomorphic automorphism groups of countable structures have metrizable universal minimal flows.

Good example

$\mathrm{Aut}(\mathbb{P})$ *is NOT oligomorphic.*

Non-metrizable universal minimal flow

Theorem (Veech)

Locally compact groups have non-metrizable universal minimal flows.

Non-metrizable universal minimal flow

Theorem (Veech)

Locally compact groups have non-metrizable universal minimal flows.

Good example

$\mathrm{Aut}(\mathbb{P})$ *is NOT locally compact.*

OBRIGADA